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Solutions of the Riccati equation and their relation to the Toda lattice

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Abstract. Continued fraction solutions to the Riccati equation are constructed using the idea of 'form invariance' of this equation under linear fractional transformation of the dependent variable. This method has recently been applied to the scalar case and a modification and extension to the corresponding matrix equation is described. It is shown that the technique is related to Euler's method for generating continued fraction solutions to the corresponding second-order linear differential equation. Several particular examples are considered and lead to some well known continued fraction representations of standard functions. It is demonstrated that the elements of the continued fraction solution are solutions of the 'Toda lattice' equations. We have therefore discovered an intriguing connection between the Riccati and the above equations which deserves further investigation.

1. Introduction

The matrix Riccati equation

$$\dot{W}(t) = A(t) + B(t)W(t) + W(t)C(t) + W(t)D(t)W(t) \tag{1.1}$$

where A, B, C, D and W are $(n \times n)$ matrix functions of t, plays a central role in many applications, e.g. optimal control theory, Bäcklund transformations for many non-linear partial differential equations, plasma physics, etc. There has been much progress recently in the understanding of how the general solution of (1.1) may be constructed by superposing particular solutions (Anderson *et al* 1983). It is a remarkable fact that in the general case only *five* particular solutions are needed to construct the general solution.

The problem is then to obtain sets of particular solutions. There have been several approaches to this problem for the scalar case and we will concentrate on that of the construction of continued fraction solutions using the idea of 'form invariance' of the Riccati equation under linear fractional transformation of the dependent variable (Chisholm 1984). As described in § 2, a simple transformation of $W(t)$ to $Z_0(t)$ leads to an equation of the 'standard form',

$$\dot{Z}_0(t) = E_0(t) + Z_0(t)F_0(t) + Z_0(t)^2 \tag{1.2}$$

where $Z_0(t)$, etc, are $(n \times n)$ matrix valued functions of t. Then we construct a continued fraction solution,

$$Z_0(t) = U_0 + [N_1(t) - U_0 - (N_2(t) - U_0 - [\ldots]^{-1}M_3(t))^{-1}M_2(t)]^{-1}M_1(t) \tag{1.3}$$

where the elements $M_k(t)$, $N_k(t)$ are constructed from the given $E_0(t)$ and $F_0(t)$, and U_0 is a constant matrix. A modified form of (1.3) which fits exactly a given initial value $Z_0(0)$ is described in the appendix.

In § 3 we demonstrate that our method is related to a matrix generalisation of a technique due to Euler (Ince 1956). Chisholm (1984) considered the scalar case where the Riccati equation can be simplified still further to the 'standard form',

$$\dot{z}_0(t) = b_0(t) - z_0(t)^2. \tag{1.4}$$

Requiring preservation of this 'form' under linear fractional transformation of z_0 is more restrictive than requiring the 'form' (1.2) to be preserved. The continued fraction solution that he obtained is therefore not identical to (1.3). However, it is equivalent to that given by Euler's method as we will demonstrate in § 3.

The elements $M_k(t)$, $N_k(t)$ of the continued fraction satisfy a pair of coupled non-linear difference equations. In § 4 we give some simple solutions of these equations and show that they give standard continued fraction expansions for the tangent and cotangent functions. These simple solutions illustrate the fact that the value of the continued fraction (1.2) changes when U_0 is changed, answering in the affirmative a question posed by Chisholm (1984).

For most examples of equation (1.2), the elements $M_k(t)$ and $N_k(t)$ soon become very complicated functions of t as k is increased. This would seem to limit the usefulness of our approach to a few special examples. However, this is not the case as we will show in § 5. There we demonstrate the remarkable fact that the equations for $M_k(t)$ and $N_k(t)$ are, in the scalar case, equivalent to the canonical equations of motion for the 'Toda lattice' (Toda 1976). These lattice equations have infinitely many solutions generated by Bäcklund transformations. For each of them we can construct a continued fraction solution of a corresponding Riccati equation. We finish § 5 by giving our conclusions on this work.

2. Continued fraction solutions to the matrix Riccati equation

It is straightforward to show that if $W(t)$ is a solution of (1.1), then

$$Z_0(t) = D(t)W(t) - K(t) \tag{2.1}$$

is a solution of the 'standard form' equation,

$$\dot{Z}_0(t) = E_0(t) + Z_0(t)F_0(t) + Z_0(t)^2 \tag{1.2}$$

when

$$K(t) = -D(t)B(t)D^{-1}(t) - \dot{D}(t)D^{-1}(t) \tag{2.2a}$$

where

$$E_0(t) = -\dot{K}(t) + D(t)A(t) + K(t)C(t) \qquad F_0(t) = C(t) + K(t). \tag{2.2b}$$

We construct the continued fraction form (1.3) for $Z_0(t)$ by defining the following sequence of linear fractional transformations:

$$Z_k(t) = U_0 + [N_{k+1}(t) - Z_{k+1}(t)]^{-1}M_{k+1}(t) \qquad k = 0, 1, 2, \ldots \tag{2.3}$$

where $N_k(t)$, $M_k(t)$ are $(n \times n)$ matrix functions of t and U_0 is an $(n \times n)$ constant matrix.

Using matrix algebra, it is easy to show that, if $Z_{k-1}(t)$ satisfies the 'standard form' equation,

$$\dot{Z}_{k-1}(t)=E_{k-1}(t)+Z_{k-1}(t)F_{k-1}(t)+Z^2_{k-1}(t) \tag{2.4}$$

then

$$\dot{Z}_k(t)=E_k(t)+Z_k(t)F_k(t)+Z^2_k(t) \qquad k=1,2,3 \tag{2.5}$$

if

$$M_k(t)=E_{k-1}(t)+U_0F_{k-1}(t)+U_0^2 \tag{2.6a}$$

and

$$N_k(t)=\dot{M}_k(t)M_k^{-1}(t)-M_k(t)[F_{k-1}(t)+U_0]M_k^{-1}(t) \tag{2.6b}$$

where

$$E_k(t)=\dot{N}_k(t)+N_k(t)U_0+M_k(t) \tag{2.7a}$$

and

$$F_k(t)=-N_k(t)-U_0. \tag{2.7b}$$

The coefficient matrices may be eliminated from (2.6) and (2.7) to give the following non-linear recurrence relations for the elements of the continued fractions (1.3) for $Z_0(t)$:

$$M_k(t)=M_{k-1}(t)+[N_{k-1}(t),\ U_0]+\dot{N}_{k-1}(t) \tag{2.8a}$$

$$N_k(t)=M_k(t)N_{k-1}(t)M_k^{-1}(t)+\dot{M}_k(t)M_k^{-1}(t). \tag{2.8b}$$

In the scalar case these relations reduce to the remarkably simple forms

$$M_k(t)-M_{k-1}(t)=\dot{N}_{k-1}(t) \tag{2.9a}$$

$$N_k(t)-N_{k-1}(t)=\dot{M}_k(t)M_k(t)^{-1} \tag{2.9b}$$

which we will return to later. Note that the 'initial' matrix functions $M_1(t)$ and $N_1(t)$ are determined from the coefficients of the original Riccati equation (1.2) by using relations (2.6) with $k=1$.

Changing U_0 will change the initial value $Z_0(0)$ as will be demonstrated through example in § 4, though there is no direct relation between them. However, it is possible to construct a continued fraction to $Z_0(t)$ so that, if it is truncated at any order, then this truncated fraction has exactly the given initial value $Z_0(0)$. The defining equations for this continued fraction are appreciably more complicated than (2.6) and (2.7) and the details are given in the appendix.

3. A comparison with Euler's method

We demonstrate first of all that Chisholm's continued fraction solution to (1.4) may be obtained using Euler's method as described by Ince (1956). The first step is to convert (1.4) to a linear second-order differential equation by making the standard substitution $z_0(t)=\dot{w}_0(t)/w_0(t)$ where now

$$\ddot{w}_0(t)-b_0(t)w_0(t)=0. \tag{3.1}$$

From this equation and its derivative,

$$z_0(t) = \dot{w}_0(t)/w_0(t) = b_0(t)\left(\frac{\dot{b}_0(t)}{2b_0(t)} + \frac{\dot{w}_1(t)}{w_1(t)}\right)^{-1} \tag{3.2}$$

where $\dot{w}_0(t) = (-b_0(t))^{1/2} w_1(t)$. The new function $w_1(t)$ satisfies an equation of similar form to (3.1), i.e.

$$\ddot{w}_1(t) - b_1(t) w_1(t) = 0 \tag{3.3}$$

where

$$b_1(t) = b_0(t) + \tfrac{3}{4}(\dot{b}_0/b_0)^2 - (\ddot{b}_0(t)/2b_0(t)). \tag{3.4}$$

Iterating, we obtain the following continued fraction solution for the Riccati equation (1.4):

$$z_0(t) = \frac{2b_0^2}{\dot{b}_0} + \frac{4b_0 b_1^2}{\dot{b}_1} + \frac{4b_1 b_2^2}{\dot{b}_2} + \dots \tag{3.5}$$

where

$$b_{r+1} = (4b_r^4 - 2\ddot{b}_r b_r^2 + 3\dot{b}_r^2 b_r)/4b_r^3. \tag{3.6}$$

This is exactly the continued fraction given in equation (2.18) of the paper by Chisholm (1984). The corresponding solution containing an arbitrary constant α given in equation (2.28) of that work is

$$z_0(t) = \alpha + \frac{(b_0 - \alpha^2)}{\frac{1}{2}\dot{b}_0(b_0 - \alpha^2)^{-1} + 2\alpha} + \frac{(b_1 - \alpha^2)}{\frac{1}{2}\dot{b}_1(b_1 - \alpha^2)^{-1} + 2\alpha} + \dots \tag{3.7}$$

where now

$$b_{r+1} = b_r + \frac{\alpha \dot{b}_r}{b_r - \alpha^2} + \frac{3\dot{b}_r^2}{4(b_r - \alpha^2)^2} - \frac{\ddot{b}_r}{2(b_r - \alpha^2)}. \tag{3.8}$$

This again may be generated using Euler's method by initially taking

$$w_0(t) = \mathrm{e}^{\alpha t} v_0(t)$$

and constructing a sequence of functions $\{v_i(t)\}$ corresponding to the sequence $\{w_i(t)\}$ taken previously.

We now show how our method is related in the general matrix case to a form of Euler's method. Our 'standard form' matrix equation (1.2) becomes, on making the substitution

$$Z_0(t) = -V_0(t)^{-1}\dot{V}_0(t) \qquad \ddot{V}_0(t) + V_0(t)E_0(t) - \dot{V}_0(t)F_0(t) = 0. \tag{3.9}$$

To construct a continued fraction solution of (1.2) containing the arbitrary constant matrix U_0, we set

$$V_0(t) = \tilde{X}_0(t)\,\mathrm{e}^{-U_0 t}$$

so that

$$\ddot{\tilde{X}}_0(t) + \tilde{X}_0(t)\tilde{e}_0(t) - \dot{\tilde{X}}_0(t)\tilde{f}_0(t) = 0 \tag{3.10}$$

where

$$e_0(t) = E_0(t) + U_0 F_0(t) + U_0^2 \qquad f_0(t) = F_0(t) + 2U_0 \tag{3.11}$$

and

$$\tilde{A} = \mathrm{e}^{-U_0 t} A\, \mathrm{e}^{U_0 t} \qquad \text{with } A = e, f.$$

We now make the substitution $\dot{\tilde{X}}_0(t) = \tilde{X}_1(t)\tilde{e}_0(t)$ and, as in the standard Euler's method, obtain the second-order differential equation for $\tilde{X}_1(t)$ by differentiating (3.10),

$$\ddot{\tilde{X}}_1(t) + \tilde{X}_1(t)\tilde{e}_1(t) - \dot{\tilde{X}}_1(t)\tilde{f}_1(t) = 0 \tag{3.12}$$

where

$$\tilde{f}_1(t) = \tilde{e}_0(t)\tilde{f}_0(t)\tilde{e}_0(t)^{-1} - \dot{\tilde{e}}_0(t)\tilde{e}_0(t)^{-1} \tag{3.13}$$

$$\tilde{e}_1(t) = \tilde{e}_0(t) - \dot{\tilde{f}}_1(t). \tag{3.14}$$

It may be proved using (3.10)-(3.13) that

$$-\tilde{X}_0(t)^{-1}\dot{\tilde{X}}_0(t) = (\tilde{n}_1(t) + \tilde{X}_1(t)^{-1}\dot{\tilde{X}}_1(t))^{-1}\tilde{m}_1(t) \tag{3.15}$$

where

$$\tilde{m}_1(t) = \tilde{e}_0(t) \qquad \tilde{n}_1(t) = -\tilde{f}_1(t). \tag{3.16}$$

Iterating, we can generate a sequence of matrix functions $\tilde{X}_k(t)$ satisfying

$$\dot{\tilde{X}}_k(t) = \tilde{X}_{k+1}(t)\tilde{e}_k(t)$$

and the differential equation

$$\ddot{\tilde{X}}_{k+1}(t) + \tilde{X}_{k+1}(t)\tilde{e}_{k+1}(t) - \dot{\tilde{X}}_{k+1}(t)\tilde{f}_{k+1}(t) = 0 \qquad k = 0, 1, 2, \ldots \tag{3.17}$$

where

$$\tilde{f}_{k+1}(t) = \tilde{e}_k(t)\tilde{f}_k(t)\tilde{e}_k(t)^{-1} - \dot{\tilde{e}}_k(t)\tilde{e}_k(t)^{-1} \tag{3.18}$$

and

$$\tilde{e}_{k+1}(t) = \tilde{e}_k(t) - \dot{\tilde{f}}_k(t). \tag{3.19}$$

Then iterating (3.15) and (3.16),

$$-\tilde{X}_0(t)^{-1}\dot{\tilde{X}}_0(t) = \{\tilde{n}_1(t) - [\tilde{n}_2(t) - (\tilde{n}_3(t)\ldots)^{-1}\tilde{m}_3(t)]^{-1}\tilde{m}_2(t)\}^{-1}\tilde{m}_1(t) \tag{3.20}$$

with

$$\tilde{m}_{k+1}(t) = \tilde{e}_k(t) = \tilde{m}_k(t) + \dot{\tilde{n}}_k(t) \tag{3.21}$$

$$\tilde{n}_{k+1}(t) = -\tilde{f}_{k+1}(t) = \tilde{m}_{k+1}(t)\tilde{n}_k(t)\tilde{m}_{k+1}(t)^{-1} + \dot{\tilde{m}}_{k+1}(t)\tilde{m}_{k+1}(t)^{-1}. \tag{3.22}$$

Therefore, from (3.20),

$$\begin{aligned} Z_0(t) &= -V_0(t)^{-1}\dot{V}_0(t) = -\mathrm{e}^{U_0 t}[\tilde{X}_0(t)^{-1}(\dot{\tilde{X}}_0(t) - \tilde{X}_0(t)U_0)]\,\mathrm{e}^{-U_0 t} \\ &= U_0 + \{n_1(t) - [n_2(t) - (n_3(t)\ldots)^{-1}m_3(t)]^{-1}m_2(t)\}^{-1}m_1(t) \end{aligned} \tag{3.23}$$

where now

$$m_k(t) = \mathrm{e}^{U_0 t}\tilde{m}_k(t)\,\mathrm{e}^{-U_0 t} \qquad n_k(t) = \mathrm{e}^{U_0 t}\tilde{n}_k(t)\,\mathrm{e}^{-U_0 t} \tag{3.24}$$

and above from (3.21) and (3.22),

$$m_{k+1}(t) = m_k(t) + \dot{n}_k(t) + [n_k(t), U_0] \tag{3.25}$$

$$n_{k+1}(t) = m_{k+1}(t)n_k(t)m_{k+1}(t)^{-1} + \dot{m}_{k+1}(t)m_{k+1}(t)^{-1} + [m_{k+1}(t), U_0]m_{k+1}(t)^{-1}. \tag{3.26}$$

Finally, setting $M_k(t)=m_k(t)$, $N_k(t)=n_k(t)+U_0$, we obtain from (3.23) the continued fraction (1.3) for the solution of the matrix Riccati equation (1.2), where from (3.25) and (3.26),

$$M_{k+1}(t)=M_k(t)+\dot{N}_k(t)+[N_k(t),\,U_0] \tag{3.27}$$

$$N_{k+1}(t)=\dot{M}_{k+1}(t)M_{k+1}(t)^{-1}+M_{k+1}(t)N_k(t)M_{k+1}{}^{-1}(t). \tag{3.28}$$

These are exactly the relations (2.8) so the matrix continued fraction is that obtained previously, demonstrating the connection between the method and that due to Euler.

Note that we have generated the sequence of matrix functions $\{\tilde{X}_k(t)\}$ by requiring $\dot{\tilde{X}}_k(t)=\tilde{X}_{k+1}(t)\tilde{e}_k(t)$ whereas, in the corresponding scalar case, Chisholm generated the sequence of scalar functions $\{w_k(t)\}$ by requiring $\dot{w}_k(t)=(-b_k(t))^{1/2}w_{k+1}(t)$. The square root in the latter case implies that the continued fraction solutions we obtain in the scalar case will *not* be equivalent to those of Chisholm.

4. Examples

In general the elements $M_k(t)$, $N_k(t)$ of the continued fraction increase rapidly in complexity as k is increased. However, the non-linear recurrence relations which determine these elements have some simple solutions. We confine ourselves to the scalar case. An obvious solution of (2.9) is

$$M_k(t)=-\alpha-(k-1) \qquad N_k(t)=-t \qquad U_0=0 \tag{4.1}$$

where α is a constant. The corresponding Riccati equation is, from (2.7),

$$\dot{Z}_0(t)=-\alpha+tZ_0(t)+Z_0(t)^2 \tag{4.2}$$

with continued fraction solution

$$Z_0(t)=\frac{\alpha}{t}+\frac{\alpha+1}{t}+\frac{\alpha+2}{t}+\ldots. \tag{4.3}$$

This continued fraction is used as an example for Euler's method by Ince (1956) and (4.3) is actually the continued fraction for

$$Z_0(t)=\int_0^\infty \exp(-tx-\tfrac{1}{2}x^2)x^\alpha\,\mathrm{d}x\left(\int_0^\infty \exp(-tx-\tfrac{1}{2}x^2)x^{\alpha-1}\,\mathrm{d}x\right)^{-1}. \tag{4.4}$$

Another simple solution of (2.9) is

$$N_k(t)=2k-1 \qquad M_k(t)=\mathrm{e}^{2t} \qquad U_0=0. \tag{4.5}$$

The corresponding Riccati equation is

$$\dot{Z}_0(t)=\mathrm{e}^{2t}+Z_0(t)+Z_0^2(t) \tag{4.6}$$

which then has the continued fraction solution

$$Z_0^{\mathrm{A}}(t)=\frac{\mathrm{e}^{2t}}{1}-\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{2t}}{5}-\ldots \tag{4.7}$$

$$=\frac{y^2}{1}-\frac{y^2}{3}-\frac{y^2}{5}-\ldots \tag{4.8}$$

where $y=\mathrm{e}^t$.

The RHS of (4.8) is the well known continued fraction expansion for $y \tan y$ so that we have obtained the solution,

$$Z_0^A(t) = e^t \tan(e^t) \tag{4.9}$$

of (4.6).

Similarly (2.9) has the solution

$$N_k(t) = 2k \qquad M_k(t) = e^{2t} \qquad U_0 = -1 \tag{4.10}$$

and the corresponding Riccati equation is again (4.6). The continued fraction solution is now

$$Z_0^B(t) = -1 + \frac{e^{2t}}{3} - \frac{e^{2t}}{5} - \frac{e^{2t}}{7} - \dots \tag{4.11}$$

Comparing (4.8) and (4.9), we see that

$$Z_0^A(t) Z_0^B(t) = -e^{2t} \tag{4.12}$$

so that

$$Z_0^B(t) = -e^t \cot(e^t). \tag{4.13}$$

In his work Chisholm (1984) queried whether different solutions of the scalar equation (3.1) would be obtained by changing the scalar constant α in the continued fraction (3.7). The example we have just considered illustrates the fact that our continued fraction solutions *do* change when the constant U_0 is changed.

Numerical studies show that the continued fractions to (4.6) converge rapidly for a wide range of values for t and different values for U_0.

5. The Toda lattice and conclusions

In § 4 we gave some simple solutions to the non-linear recurrence relations for the elements of our continued fraction solutions. These solutions were very special and, at first sight, it would seem that there are few cases where (2.9) would have relatively simple solutions.

We show that this is not the case by demonstrating that the non-linear relations (2.9) are equivalent to the canonical equations for the 'Toda lattice'. This lattice consists of a one-dimensional chain of particles of unit mass which interact with their nearest neighbours via an exponential potential (Toda 1976). Bäcklund transformations may be used to generate whole families of solutions for these lattice equations, as reviewed by Rogers and Shadwick (1982).

The lattice equations have the canonical form

$$\dot{q}(k, t) = p(k, t) \tag{5.1}$$

$$\dot{p}(k, t) = \exp\{-[q(k, t) - q(k-1, t)]\} - \exp\{-[q(k+1, t) - q(k, t)]\} \tag{5.2}$$

where $q(k, t)$ is the coordinate of the kth particle and $p(k, t)$ is its conjugate momentum. Setting $N_k(t) \equiv -p(k, t)$ and $M_k(t) \equiv \exp\{-[q(k, t) - q(k-1, t)]\}$ the lattice equation (5.2) may be written as

$$\dot{N}_k(t) = M_{k+1}(t) - M_k(t) \tag{5.3}$$

and on differentiating the defining equation for $M_k(t)$ and using (5.1),

$$\dot{M}_{k+1}(t)=M_{k+1}(t)[N_{k+1}(t)-N_k(t)]. \tag{5.4}$$

It is easy to see that after some minor rearrangement (5.3) and (5.4) are equivalent to the non-linear equation (2.9) for the elements of our continued fraction solutions. Therefore, to each of the infinitely many solutions of the Toda lattice equations, there is an explicit continued fraction solution of a corresponding scalar Riccati equation.

For instance, we can take the 'soliton' solution

$$M_k(t)=1+\beta^2\operatorname{sech}^2(\alpha k+\beta t) \qquad k=0,1,2,\ldots \tag{5.5}$$

where $\beta=\sinh\alpha$ with α a real constant. (5.3) may be integrated to give

$$N_k(t)=\beta\{\tanh[\alpha(k+1)+\beta t]-\tanh[\alpha k+\beta t]\} \tag{5.6}$$

and it is easy to show, by substitution, that $M_k(t)$ and $N_k(t)$ do satisfy the non-linear recurrence relations (2.9). The functions $E_0(t)$, $F_0(t)$ in the related Riccati equation are easily constructed using (2.7) but are quite complicated.

As another example, we may consider the periodic solution, the so-called 'cnoidal wave'. We have

$$M_k(t)=1+\beta^2[\operatorname{dn}^2(\alpha k+\beta t)-(E/K)] \qquad k=0,1,2,\ldots \tag{5.7}$$

where

$$\frac{1}{\beta^2}=\frac{1}{\operatorname{sn}^2(\alpha k)}+\frac{E}{K}-1$$

in which dn and sn are Jacobian elliptic functions, K and E being the complete elliptic integrals of the first and second kind respectively. We then obtain

$$N_k(t)=\beta\{Z[\alpha(k+1)+\beta t]-Z[\alpha k+\beta t]\} \tag{5.8}$$

in which Z is the Jacobian zeta function. As for the 'soliton' case, $N_k(t)$, $M_k(t)$ may be shown to satisfy the relations (2.9) and $E_0(t)$, $F_0(t)$ in the Riccati equation may be formed.

The corresponding continued fractions, derived using (2.3), are closely related to J fractions, which have been studied extensively, e.g. Jones and Thron (1980).

We have continued the study by Chisholm (1984) into the construction of continued fraction solutions to the Riccati equation using 'form invariance' of the equation under linear fractional transformation of the dependent variable. We have demonstrated that the technique can be extended to the matrix case and that there is a close relation to the method of Euler for constructing continued fractions for the logarithmic derivative of the corresponding linear second-order differential equation. In general, the elements of the continued fraction rapidly become very complicated functions of t, especially in the matrix case. We intend to study in a future work alternative methods for developing continued fraction solutions to the matrix Riccati equation.

What may well prove to be the most interesting result of this work is the demonstration of a connection between the 'Toda lattice' equations and the Riccati equation. We think this intriguing connection between two important sets of non-linear differential equations of mathematical physics merits further study.

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Appendix

There is a unique solution of the Riccati equation (1.2) for a given initial value $Z_0(0)$. We show how to construct a continued fraction solution which gives exactly $Z_0(0)$ when it is truncated at any order.

We start by making the linear fractional transformation

$$Z_0(t) = U_0 + V_0 t^2 + (N_1(t) - Z_1(t))^{-1} M_1(t) \tag{A1}$$

where both U_0, V_0 are independent of t.

Then, substituting in (1.2), we find that

$$\dot{Z}_1(t) = E_1(t) + Z_1(t) F_1(t) + Z_1(t)^2 \tag{A2}$$

if

$$M_1(t) = E_0(t) + (U_0 + V_0 t^2) F_0(t) + (U_0 + V_0 t^2)^2 - 2V_0 t \tag{A3}$$

$$N_1(t) = \dot{M}_1(t) M_1(t)^{-1} - M_1(t)(F_0(t) + U_0 + V_0 t^2) M_1(t)^{-1}. \tag{A4}$$

The new coefficient matrices are

$$F_1(t) = -N_1(t) - U_0 - V_0 t^2 \tag{A5}$$

$$E_1(t) = \dot{N}_1(t) + N_1(t)(U_0 + V_0 t^2) + M_1(t). \tag{A6}$$

To ensure that the truncated continued fraction is exact at $t = 0$ at any order, we set $Z_1(0) = 0$. From (A1) this will be the case if

$$U_0 = Z_0(0) - 1 \qquad V_0 = \tfrac{1}{2}[\dot{E}_0(0) + U_0 \dot{F}_0(0) - M_1(0)^2 - M_1(0)(F_1(0) + U_0)]. \tag{A7}$$

Note that the factor t^2 multiplying V_0 ensures that $M_1(0)$ does not depend on V_0 though $M_1(t)$ does at general time. Therefore (A7) determines U_0, V_0 in terms of $Z_0(0)$, $E_0(0)$ and $F_0(0)$.

Iterating, we have

$$Z_k(t) = U_k + V_k t^2 + (N_{k+1}(t) - Z_{k+1}(t))^{-1} M_{k+1}(t) \tag{A8}$$

with

$$M_{k+1}(t) = E_k(t) + (U_k + V_k t^2) F_k(t) + (U_k + V_k t^2)^2 - 2V_k t \tag{A9}$$

$$N_{k+1}(t) = \dot{M}_{k+1}(t) M_{k+1}(t)^{-1} - M_{k+1}(t)(F_k(t) + U_k + V_k t^2) M_{k+1}(t)^{-1} \tag{A10}$$

and

$$F_{k+1}(t) = -N_{k+1}(t) - U_k - V_k t^2 \tag{A11}$$

$$E_{k+1}(t) = \dot{N}_{k+1}(t) + N_{k+1}(t)(U_k + V_k t^2) + M_{k+1}(t). \tag{A12}$$

The 'initial condition' matrices are given by

$$U_k = -1 + \delta_{k,0} Z_0(0) \tag{A13}$$

$$V_k = \tfrac{1}{2}[\dot{E}_k(0) + U_k \dot{F}_k(0) - M_{k+1}(0)^2 - M_{k+1}(F_k(0) + U_k)]. \tag{A14}$$

Once again it should be noted that $M_{k+1}(0)$ does not depend on V_k though $M_{k+1}(t)$ does, so (A14) does give V_k. As in the former case the coefficient matrices may be eliminated from (A9)-(A14) to give

$$M_{k+1}(t)=\dot{N}_k(t)+N_k(t)(U_{k-1}+V_{k-1}t^2)+M_k(t)-(U_k+V_kt^2)(N_k(t)+U_{k-1}+V_{k-1}t^2)$$
$$+(U_k+V_kt^2)^2-2V_kt \tag{A15}$$

$$N_{k+1}(t)=\dot{M}_{k+1}(t)M_{k+1}(t)^{-1}-M_{k+1}(t)(-N_k(t)-U_{k-1}-V_{k-1}t^2+U_k+V_kt^2)M_{k+1}(t)^{-1}. \tag{A16}$$

These non-linear recurrence relations have the merit of giving a continued fraction which fits a given initial value $Z_0(0)$ when it is truncated at any order. However, they are probably too complicated to use in practice.

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